Linear programming formation for Critical Path Method

Imagine a project, which consists of multiple activities. Each activity can be executed only after its predecessor activities are all completed and the project will be finished when its activities are all completed.

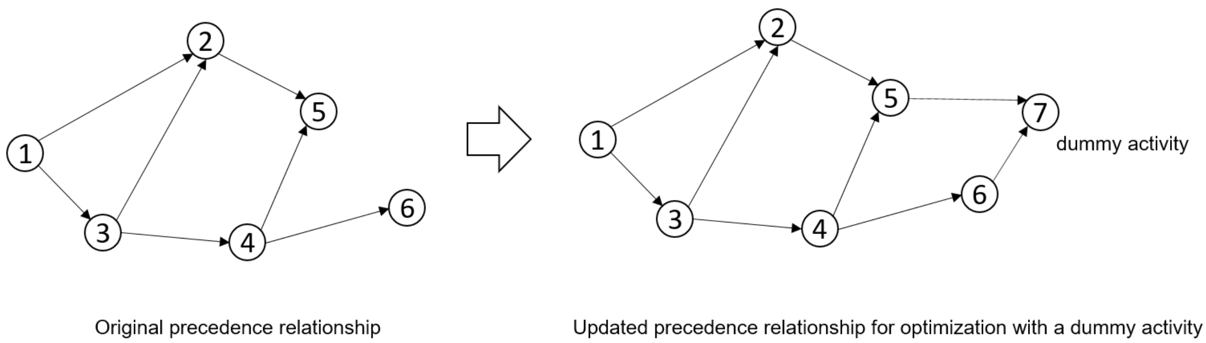
Our goal here as a project manager is to calculate the minimum duration of the project (or when the project can be completed at the earliest). Note that the duration of the project is determined by a *critical path* of the project that is a sequence of activities with the longest duration.

Let us formally describe the critical path finding problem. There are N numbers of activities for a project, indexed by i. For the simplicity, the activity with no predecessor (the activity that can immediately be started) is indexed by 1. Activity 1 can be started at time 0. If multiple activities have no predecessors, any activity of those can be the activity 1.

Activity i takes t_i time units to be completed. If activity i should be completed to start activity j, this precedence relationship is denoted by a pair of the activities (i, j). All the precedence relationships for the project are stored in set A.

The decision variable of this problem is the starting time of activity i, denoted by X_i, $\forall i \in \{1, \ldots, N+1\}$. Please note the additional index $N+1$ that is for a dummy activity. The dummy activity $N+1$ is introduced to calculate the duration of the project. Specifically, with the dummy activity, whose precedence relationships are associated with the activities with no successor in the project, the starting time of the dummy activity can represent the project completion.

The concepts of the dummy activity and corresponding updates on the precedence relationship are illustrated in the figure below, where a node represents an activity and an arc between nodes represents the precedence relationship.



With the setting, the objective function of the problem can be written by

$$\min X_{N+1} - X_1,$$

which is to minimize the gap between the ending and starting time of the project, i.e. the project duration. This objective function is then minimized subject to the following constraints:

$$X_i + t_i \leq X_j \quad \forall (i, j) \in A,$$

$$X_i \geq 0 \quad \forall i \in \{1, \ldots, N+1\}.$$

The first constraint ensures that the starting time of an activity (X_j) must be greater than or equal to the starting time of its predecessor activity (X_i) plus the duration of the predecessor activity (t_i). The last constraint is the positive value constraint.