Design and analysis of experiments
Lecture 9

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Fixed, random and mixed factors

- Last time we saw two-way ANOVA with two fixed factors - but the factors could also be random, or there could be one of each.
- A mixed model contains both random and fixed factors.

Two-way ANOVA with fixed, random or mixed factors

- The model:

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$$

  where $i = 1, \ldots, a$, $j = 1, \ldots, b$, $k = 1, \ldots, n$ (balanced case)

- Assumptions:

	Fixed factors	Random factors	Mixed factors
A	$\sum_i \alpha_i = 0$	$\alpha_i \sim N(0, \sigma_A^2)$	$\sum_i \alpha_i = 0$
B	$\sum_j \beta_j = 0$	$\beta_j \sim N(0, \sigma_B^2)$	$\beta_j \sim N(0, \sigma_B^2)$
AB	$\sum_* (\alpha\beta)_{ij} = 0$	$(\alpha\beta)_{ij} \sim N(0, \sigma_{AB}^2)$	$(\alpha\beta)_{ij} \sim N(0, \sigma_{AB}^2)$
E	$\epsilon_{ijk} \sim N(0, \sigma_E^2)$	$\epsilon_{ijk} \sim N(0, \sigma_E^2)$	$\epsilon_{ijk} \sim N(0, \sigma_E^2)$

- $* = i$ or j
- Note that when combining a fixed and a random factor, we get a random interaction term.
- The mixed model given here is called unrestricted, since all random variables are independent; in the restricted case $\sum_i (\alpha\beta)_{ij} = 0$. We use the unrestricted case.

Expected mean squares and variance components

- The expected mean squares for the three designs:

	Fixed factors	Random factors	Mixed factors
$\mathbb{E}[MS_A]$	$\sigma_E^2 + \frac{bn\sum\alpha_i^2}{a-1}$	$\sigma_E^2 + n\sigma_{AB}^2 + bn\sigma_A^2$	$\sigma_E^2 + n\sigma_{AB}^2 + \frac{bn\sum\alpha_i^2}{a-1}$
$\mathbb{E}[MS_B]$	$\sigma_E^2 + \frac{an\sum\beta_j^2}{b-1}$	$\sigma_E^2 + n\sigma_{AB}^2 + bn\sigma_B^2$	$\sigma_E^2 + n\sigma_{AB}^2 + bn\sigma_B^2$
$\mathbb{E}[MS_{AB}]$	$\sigma_E^2 + \frac{n\sum(\alpha\beta)_{ij}^2}{(a-1)(b-1)}$	$\sigma_E^2 + n\sigma_{AB}^2$	$\sigma_E^2 + n\sigma_{AB}^2$
$\mathbb{E}[MS_E]$	σ_E^2	σ_E^2	σ_E^2

- From this variance components can be estimated.

Hypothesis

- In all three versions we can test whether the main effects and interaction are significant - for fixed factors/interactions, we are testing whether the means are 0, and for random factors/interactions, we are testing whether the variance component is 0.
- Beware: in the random/mixed model the test statistic changes for the main factors:
  - Fixed model: $F_A = \frac{MS_A}{MS_E}$
  - Random/mixed model: $F_A = \frac{MS_A}{MS_{AB}}$

  Notice that we always compare with the interaction if it is random, otherwise with error term. The degrees of freedom changes accordingly. This only holds for the unrestricted mixed model, not the restricted one.

R

- R demo, Part 1
- Exercise 1

Three-way ANOVA

- If we have three factors we can do three-way ANOVA - obviously we can extent ANOVA to any number of factors, but we will stop at three.
- If you need much more than three factors in your experiment, then maybe you should reconsider your experiment.
- Example 5.3, p. 208, a soft drink bottler is trying to get a uniform fill height in bottles. This can potentially depend on three factors:

  A Percent carbonation

  B Operating pressure

  C Bottles produced per minute

  All of these can interact in various ways.

Interaction in three-way ANOVA

- Two-factor interaction:
  - There are now three two-factor interactions: AB, AC and BC.
  - The interpretation is the same as for the two-factor case, i.e. if A and B interacts, then the effect of A changes depending on the level of B (or vice versa).
  - Example: Two companies (A) may have different rules about monthly payment for the two genders (B) - one has strict equality, while the other does not.
- Three-factor interaction:
  - There is only one three-way interaction, ABC.
  - Three-way interaction means that one or more two-way interactions differ across the levels of the third factor.
  - Example: Two companies (A) may have different rules about monthly payment to the two genders (B) for various positions (C). For example, one company may have strict rules about equality for all positions, while the other have much more loose rules, which by incidence only influence the monthly payment for certain positions.
- Interaction between four or more factors gets complicated.

Three-way ANOVA

- The model with factors A, B and C:

  $$y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijk}$$

  where $i = 1, \ldots, a$, $j = 1, \ldots, b$, $k = 1, \ldots, c$, $l = 1, \ldots, n$ (balanced case)
- The constraints, sums of squares, degrees of freedom, F-test statistics, etc. can be found in the same way as for simpler ANOVA - we will skip this.

R

- R-demo, part 2
- Exercise 2
- R-demo, part 3 - response surfaces
- Exercise 3