Design and analysis of experiments
Lecture 5

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Fixed vs. random factors

- We will look at a variant of ANOVA with random effects.
- Fixed effects: we compare a selection of treatments.
- Random effects: we compare a large selection of treatments (possibly infinite) by taking a random sample.

Examples

- Fixed effects example: Compare a small number of different machines to see if they produce the same output. Here we are interested in the particular machines chosen.
- Random effect example: Compare all the machines in a factory by taking a random sample, and comparing these. The conclusion applies to all the machines, not the particular ones we have chosen.
- Compare the etch rate data (Table 3.1, p. 67) with the dataset in Example 3.11 (p. 119) - the data looks the same, but the idea is different.
- When setting up an experiment, the reason for choosing between a random and a fixed factor should be practical, not statistical.

ANOVA with random effects

- Model:

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij}, \qquad i = 1, \ldots, a, \quad j = 1, \ldots, n$$

- Both α_i and ϵ_{ij} are random now:

$$\alpha_i \sim N(0, \sigma_A^2) \qquad \epsilon_{ij} \sim N(0, \sigma_E^2)$$

  (In Montgomery $\sigma_A^2 = \sigma_\tau^2$ and $\sigma_E^2 = \sigma^2$)
- A is called a random factor now (versus a fixed factor)
- Note that we are looking at the balanced case now.
- The hypothesis that we test is that there is no effect from the random factor, i.e. its variance is zero:

$$H_0 : \sigma_A^2 = 0$$
$$H_1 : \sigma_A^2 > 0$$

Sums of squares

- Model:

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij}$$

- Means:

$$\bar{y}_{i\bullet} = \mu + \alpha_i + \bar{\epsilon}_{i\bullet}$$
$$\bar{y}_{\bullet\bullet} = \mu + \alpha_\bullet + \bar{\epsilon}_{\bullet\bullet}$$

- Sum of squares, error:

$$SS_E = \sum_i \sum_j (y_{ij} - \bar{y}_{i\bullet})^2 = \sum_i \sum_j (\epsilon_{ij} - \bar{\epsilon}_{i\bullet})^2 \sim \sigma_E^2 \chi^2_{N-a}$$

- This is the same as in the fixed factor case.

Sums of squares

- Sum of squares, factor A:

$$\begin{aligned} SS_A &= \sum_i \sum_j (\bar{y}_{i\bullet} - \bar{y}_{\bullet\bullet})^2 = n \sum_i ((\alpha_{i\bullet} + \bar{\epsilon}_{i\bullet}) - (\alpha_{\bullet} + \bar{\epsilon}_{\bullet\bullet}))^2 \\ &\sim n(\sigma_A^2 + \sigma_E^2/n)\chi^2_{a-1} \end{aligned}$$

since

$$\mathrm{Var}(\alpha_{i\bullet} + \bar{\epsilon}_{i\bullet}) = \sigma_A^2 + \sigma_E^2/n$$

- This is different than the fixed factor case.

Mean sums of squares and test statistic

- Mean sums of squares

$$MS_E \sim \sigma_E^2 \frac{\chi^2_{N-a}}{N-a} \qquad MS_A \sim (\sigma_E^2 + n\sigma_A^2)\frac{\chi^2_{a-1}}{a-1}$$

- Test statistic:

$$F = \frac{MS_A}{MS_E} \sim \left(1 + \frac{n\sigma_A^2}{\sigma_E^2}\right) F_{a-1,N-a}$$

- Under H_0 we have that $\sigma_A^2 = 0$:

$$F_0 = \frac{MS_A}{MS_E} \sim F_{a-1,N-a}$$

- We reject H_0 if $F_0 > F_{a-1,N-a;\alpha}$
- Notice that the test is the same as in the fixed effect case.

Variance components

- The variance components σ_A^2 and σ_E^2 are a way of quantifying where variability comes from in a dataset - we need to find estimates $\hat{\sigma}_A^2$ and $\hat{\sigma}_E^2$ for these.
- Expected mean sums of squares:

$$\mathbb{E}[MS_E] = \sigma_E^2, \qquad \mathbb{E}[MS_A] = \sigma_E^2 + n\sigma_A^2$$

- Estimates of σ_A^2 and σ_E^2:

$$\hat{\sigma}_E^2 = MS_E,$$
$$\hat{\sigma}_E^2 + n\hat{\sigma}_A^2 = MS_A \Rightarrow \hat{\sigma}_A^2 = \frac{MS_A - MS_E}{n}$$

- Note that $\hat{\sigma}_A^2$ can be negative!

Confidence intervals

- Confidence interval for σ_E^2:

$$\left[\frac{SS_E}{\chi_{\nu_E,\alpha/2}}, \frac{SS_E}{\chi_{\nu_E,1-\alpha/2}}\right]$$

- No exact confidence interval for σ_A^2, but we can find the confidence interval for $\sigma_A^2/(\sigma_A^2+\sigma_E^2)$:

$$\left[\frac{\phi_{\nu_A,\nu_E,\alpha/2}}{1+\phi_{\nu_A,\nu_E,\alpha/2}}, \frac{\phi_{\nu_A,\nu_E,1-\alpha/2}}{1+\phi_{\nu_A,\nu_E,1-\alpha/2}}\right]$$

where

$$\phi_{\nu_A,\nu_E,\alpha} = \frac{1}{n}\left(\frac{MS_A}{MS_E F_{\nu_A,\nu_E;\alpha}} - 1\right)$$

R

- R demo, part 1
- Exercise 1

Randomised complete block design

- Sometimes we are interested in testing whether a treatment (factor A) has an effect, but another factor (B) also influences the experiment
- For example, we may be interested in differences between a number of machines, but the operator using the machine may influence the results. An experiment could then be to choose some operators (at random) and let them try each machine.
- This is closely related to the paired t-test.
- In this lecture we assume that A is a fixed factor, and B is a random factor - Montgomery uses two fixed factors.

Randomised complete block design

- Randomised: within each block the order of measurements should be randomised.
- Complete: all combinations of levels in A and B should be measured.
- Block: B is a block and is typically a nuisance factor, i.e. we are not interested in whether the influence of this is significant or not.

RCBD - the model

- Model:

$$y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}, \qquad i = 1, \ldots, a, \quad j = 1, \ldots, b$$

- Note that μ and α_i are fixed constants, while β_j and ϵ_{ij} are random variables.
- Assumptions:

$$\sum_i \alpha_i = 0, \qquad \beta_j \sim N(0, \sigma_B^2), \qquad \epsilon_{ij} \sim N(0, \sigma_E^2)$$

All β_j and ϵ_{ij} are assumed independent.

Sums of squares

- Model:

$$y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}$$

- Means:

$$\begin{aligned}
\bar{y}_{i\bullet} &= \mu + \alpha_i + \bar{\beta}_{\bullet} + \bar{\epsilon}_{i\bullet} \\
\bar{y}_{\bullet j} &= \mu + \beta_j + \bar{\epsilon}_{\bullet j} \\
\bar{y}_{\bullet\bullet} &= \mu + \bar{\beta}_{\bullet} + \epsilon_{\bullet\bullet}
\end{aligned}$$

- Sums of squares:

$$\begin{aligned}
SS_A &= a \sum b_j^2 \qquad & b_j &= \bar{y}_{\bullet j} - \bar{y}_{\bullet\bullet} = (\beta_j + \bar{\epsilon}_{\bullet j}) - (\bar{\beta}_{\bullet} + \bar{\epsilon}_{\bullet\bullet}) \\
SS_B &= b \sum a_i^2 \qquad & a_i &= \bar{y}_{i\bullet} - \bar{y}_{\bullet\bullet} = \alpha_i + \bar{\epsilon}_{i\bullet} - \bar{\epsilon}_{\bullet\bullet} \\
SS_E &= \sum_{i,j} e_{ij}^2 \qquad & e_{ij} &= y_{ij} - \bar{y}_{i\bullet} - \bar{y}_{\bullet j} + \bar{y}_{\bullet\bullet} = \epsilon_{ij} - \bar{\epsilon}_{i\bullet} - \bar{\epsilon}_{\bullet j} + \bar{\epsilon}_{\bullet\bullet}
\end{aligned}$$

Distributions of sums of squares

- Distributions:

$$
\begin{aligned}
SS_A &\sim \sigma_E^2 \chi^2_{\nu_A} \\
SS_B &\sim (\sigma_E^2 + a\sigma_B^2)\chi^2_{\nu_B} \\
SS_E &\sim \sigma_E^2 \chi^2_{\nu_E}
\end{aligned}
$$

Hypothesis and test of A

- Hypothesis on the fixed factor:

$$H_0 : \alpha_1 = \ldots = \alpha_a = 0$$
$$H_1 : \alpha_i \text{ not all } 0$$

- Test statistic under H_0:

$$F_A = \frac{MS_A}{MS_E} \sim F_{\nu_A, \nu_E}$$

- Reject H_0 if $F_A > F_{\nu_A, \nu_E; \alpha}$

Hypothesis and test of B

- Hypothesis on the random factor:

$$H_0 : \sigma_B^2 = 0$$
$$H_1 : \sigma_B^2 > 0$$

- Test statistic under H_0:

$$F_B = \frac{MS_B}{MS_E} \sim \left(1 + \frac{a\sigma_B^2}{\sigma_E^2}\right) F_{\nu_B,\nu_E} = F_{\nu_B,\nu_E}$$

- Reject H_0 if $F_B > F_{\nu_B,\nu_E;\alpha}$
- Note that we usually do not need this test in RCBD, since B is just a nuisance variable.

The ANOVA table

- The ANOVA table can be extended to this design.

Source	SS	df	MS	F	p
Treatment	SS_A	$a-1$	MS_A	F_A	p_A
Block	SS_B	$b-1$	MS_B	(F_B)	(p_B)
Error	SS_E	$(a-1)(b-1)$	MS_E		
Total	SS_{Tot}	$N-1$			

- Here $SS_{Tot} = SS_A + SS_B + SS_E$ and $N-1 = (a-1)+(b-1)+(a-1)(b-1)$

Variance components

- Expected mean sums of squares:

$$\begin{aligned}
\mathbb{E}[MS_A] &= \sigma_E^2 + \frac{b\sum \alpha_i^2}{a-1} \\
\mathbb{E}[MS_B] &= \sigma_E^2 + a\sigma_B^2 \\
\mathbb{E}[MS_E] &= \sigma_E^2
\end{aligned}$$

- Estimated variance components:

$$\begin{aligned}
\hat{\sigma}_E^2 &= MS_E \\
\hat{\sigma}_B^2 &= \frac{MS_B - MS_E}{a}
\end{aligned}$$

- Note that these are almost the same formulas as in the ANOVA with a single random factor.

R

- R-demo, part 2
- Exercise 2

Latin squares design

- Sometimes we have more than one nuisance factor, fx when we compare a number of machines, there may be different operators and materials that may influence the performance.
- It may be very time/resource-consuming to do all combinations of both nuisance factors on each level in the main factor.
- Instead we can make sure that each level in either nuisance factor is tried out in the main factor - fx each machine is tested by each operator and with each material.
- A latin square is one such design.

A visually intuitive example

- Consider a field where we need to try out 4 breeds of barley, but we are worried that the placement in north-south direction and east-west direction will influence the result - we need to place all types of barley "all over the field".
- We divide the field into 4×4 and make sure that each type of barley is placed at all north-south locations and all east-west locations, fx:

A	B	C	D
B	A	D	C
C	D	A	B
D	C	B	A

- There are 576 of 4×4 latin squares, and the number grows superexponenetially with the size - R can help you generate such designs.

Latin square designs

- There has to be the same number of levels the main factor, row factor and column factor, say p - we have to design the experiment to fulfill this.
- In a latin square design we have to make p^2 number of experiments, compared to p^3 if we wanted to do all combinations.

Latin square designs

- Model:
$$y_{ij} = \mu + \alpha_i + \beta_j + \tau_k + \epsilon_{ij}$$
where $i = 1, \dots, p$, $j = 1, \dots, p$, and $\epsilon_{ij} \sim N(0, \sigma_E^2)$
- α is the row effect, β is the column effect, and τ is the treatment effect.
- Note that $k = k(i, j)$ is specified by i and j and therefore omitted in y_{ijk}.
- Formulas and ANOVA table are given in ch. 4.2 in Montgomery - we just use R with GAD package:
  - If all factors are fixed (as in the book):
    `gad(y~as.fixed(R)+as.fixed(C)+as.fixed(T))`
    Here we assume $\sum \alpha_i = \sum \beta_j = \sum \tau_k = 0$
  - If only the main effect is fixed:
    `aov(y~as.random(R)+as.random(C)+as.fixed(T))`
    Here we assume $\alpha_i \sim N(0, \sigma_A^2)$, $\beta_j \sim N(0, \sigma_B^2)$, and $\sum \tau_k = 0$
- Multiple generalizations to replicated latin squares exist, if we are not satisfied with only one measurement of each combination - see the book for details.

R

- R-demo, part 3